◊ **Exercice 1. A function chain complex.** Just like categories are enriched over themselves (there is a category of functors and natural transformations), and just like spaces are enriched over themselves, so are chain complexes. We define in this exercise a chain complex of chain maps. Let $C_\bullet, D_\bullet$ be two chain complexes of R-modules.
Define $\mathscr{H}om_R(C_\bullet, D_\bullet)_n$ to be the product $\prod_q \mathrm{Hom}_R(C_q, D_{q+n})$ and the differential is given by $d_n(f) = d^D_{n+q} \circ f + (-1)^n f \circ d^C_q$.
Recall that the tensor product $C_\bullet \otimes D_\bullet$ is given in degree n by $\bigoplus_{i+j=n} C_i \otimes D_j$ and the differential is given on the component $C_i \otimes D_j$ by $d^C_i \otimes id + (-1)^i id \otimes d^D_j$.

1. Verify that this defines a chain complex $\mathscr{H}om_R(C_\bullet, D_\bullet)$.
2. Show that $H_0(\mathscr{H}om_R(C_\bullet, D_\bullet)) \cong [C_\bullet, D_\bullet]$, the abelian group of homotopy classes of chain maps.
3. Show that $H_n(\mathscr{H}om_R(C_\bullet, D_\bullet)) \cong [C_\bullet[n], D_\bullet]$, the abelian group of homotopy classes of chain maps of degree n.
4. Verify that the pairing $\langle -, - \rangle \colon \mathscr{H}om_R(C_\bullet, D_\bullet) \otimes_R C_\bullet \to D_\bullet$ defined by $\langle f, c \rangle = f(c)$ is a chain map, i.e. $d^D \langle f, c \rangle = \langle df, c \rangle + (-1)^n \langle f, d^C c \rangle$.
5. Prove that composition $\mathscr{H}om_R(D_\bullet, E_\bullet) \otimes \mathscr{H}om_R(C_\bullet, D_\bullet) \to \mathscr{H}om_R(C_\bullet, E_\bullet)$ is a chain map (use the notation from point 4.).

Solution 1 (by Blaise and Berk)

1. First, $Hom_R(C_\bullet, D_\bullet)_n$ are R-modules and d is R-linear. We show that $d^2 = 0$. For $f \in Hom_R(C_\bullet, D_\bullet)_n$ let us denote by f_i the map from C_i to D_{i+n}. Then we compute $d_{n+1}(f_i) = d^D_{n+1+i}(f_i) - (-1)^n f_{i-1} \circ d^C_i$. Then composing with d_n we get,
$$\begin{aligned} d_n \circ d_{n+1}(f_i) = d^D_{n+i} \circ d^D_{n+1+i} f_i - (-1)^{n+1} d^D_{n+i} f_{i-1} \circ d^C_i \\ - (-1)^n d^D_{n+i} f_{i-1} \circ d^C_i + (-1)^{2n+1} f_{i-2} \circ d^C_{i-1} \circ d^C_i. \end{aligned}$$
Using that $(d^D)^2 = 0 = (d^C)^2$ we see that equation above is equal to zero.
2. Elements of $Hom_R(C_\bullet, D_\bullet)_0$ are collections of maps $f_q : C_q \to D_q$. They are in general not chain maps as they do not commute with the differential. However, cycles are exactly the collection of such collections which are chain maps.
Indeed, being in the kernel of d means that $d(f)_q = 0 = d^D_q(f_q) - f_q(d^C_q)$ which is exactly the condition needed for being a chain map.

   So $H_0(Hom_R(C_\bullet, D_\bullet))$ is a quotient of the abelian group of chain maps. So it suffices to show that this quotient is exactly the same as the equivalence relation given by homotopy.

   As $H_0(Hom_R(C_\bullet, D_\bullet)) = ker(d_0)/Im(d_1)$, it's enough to show that $f \simeq 0 \iff f \in Im(d_1)$

To show this, we can remark that $f \simeq 0$ means that there is a homotopy $h_q : C_{q-1} \to D_q$ such that $f_q - 0 = h_q \circ d_q^C + d_{q+1}^D \circ h_{q+1} = d(h)_q$. As $h_q : C_{q-1} \to D_q$, it is an element of $Hom_R(C_\bullet, D_\bullet)_1$.
We now see that the condition $f \simeq 0$ through h is equivalent to $d(h) = f$, so our claim is proved.

3. Now we want to show that $\mathrm{H}_n(\mathscr{H}om_R(C_\bullet, D_\bullet)) \cong [C_\bullet[n], D_\bullet]$:

First we observe that every $f \in \prod_q \mathrm{Hom}_R(C_q, D_{q+n})$ can be seen as an element of

$$\prod_q \mathrm{Hom}_R(C[n]_{q+n}, D_{q+n}),$$

indeed : $\forall q$ we write

$$F_{q+n} = f_q : C[n]_{q+n} = C_q \longrightarrow D_{q+n}$$

Then F is in $\prod_q \mathrm{Hom}_R(C[n]_q, D_q)$ and is uniquely defined by f so we can define the following isomorphism :

$$\Phi : \prod_q \mathrm{Hom}_R(C_q, D_{q+n}) \longrightarrow \prod_q \mathrm{Hom}_R(C[n]_q, D_q)$$
$$f \longmapsto F$$

By point 2) we have that $\mathrm{H}_0(\mathscr{H}om_R(C_\bullet[n], D_\bullet)) \cong [C_\bullet[n], D_\bullet]$. Let us proove that Φ defines an isomorphism from $\mathrm{H}_n(\mathscr{H}om_R(C_\bullet, D_\bullet))$ to $\mathrm{H}_0(\mathscr{H}om_R(C_\bullet[n], D_\bullet))$.

First f is a chain map of degree n if and only if F is a chain map because those two commutative diagrams are equivalent :

$$\begin{array}{ccc} C[n]_{n+q} & \xrightarrow{F_{n+q}} & D_{q+n} \\ \downarrow d^{C[n]}_{n+q} & & \downarrow d^D_{q+n} \\ C[n]_{n+q-1} & \xrightarrow{F_{n+q-1}} & D_{q+n-1} \end{array} \iff \begin{array}{ccc} C_q & \xrightarrow{f_q} & D_{q+n} \\ \downarrow (-1)^n d^C_q & & \downarrow d^D_{q+n} \\ C_{q-1} & \xrightarrow{f_{q-1}} & D_{q+n-1} \end{array}$$

So Φ defines an isomorphism from $Z_n(\mathscr{H}om_R(C_\bullet, D_\bullet))$ to $Z_0(\mathscr{H}om_R(C_\bullet[n], D_\bullet))$ by restricting him to those groups .

Now take $f, g \in Z_n(\mathscr{H}om_R(C_\bullet, D_\bullet))$ such that F and G are homotopic equivalent when F and G are the image of f and g . Then we can find $H \in \prod_q \mathrm{Hom}_R(C[n]_{n+q}, D_{n+q+1})$ such that :

$$F_{n+q} - G_{n+q} = d^D_{n+q+1} \circ H_{q+n} + H_{n+q-1} \circ d^{C[n]}_{q+n}$$

We define h to be a map in $\prod_q \mathrm{Hom}_R(C_q, D_{q+n+1})$ by $h_q = H_{q+n}$ Then we have :

$$\begin{aligned}(d_{n+1}(h))_q &= d^D_{n+q+1} \circ h_q - (-1)^{n+1} h_{q-1} \circ d^C_q = d^D_{n+q+1} \circ H_{q+n} + H_{n+q-1} \circ d^{C[n]}_{q+n} \\ &= F_{q+n} - G_{q+n} = f_q - g_q\end{aligned}$$

Which means that f and g have the same image in $\mathrm{H}_n(\mathscr{H}om_R(C_\bullet, D_\bullet))$.

Conversely, if $f, g \in Z_n(\mathscr{H}om_R(C_\bullet, D_\bullet))$ have the same image in $\mathrm{H}_n(\mathscr{H}om_R(C_\bullet, D_\bullet))$ then we can find $h \in \prod_q \mathrm{Hom}_R(C_q, D_{q+n})$ such that $f_q - g_q = (d_{n+1}(h))_q$.
Then define H a map in $\prod_q \mathrm{Hom}_R(C[n]_{n+q}, D_{q+n+1})$ by : $H_{q+n} = h_q$. Then H gives us a homotopy between F and G.
We define π and q to be the following quotient maps :

$$\begin{aligned} \pi : Z_0(\mathscr{H}om_R(C_\bullet[n], D_\bullet)) &\longrightarrow \mathrm{H}_0(\mathscr{H}om_R(C_\bullet[n], D_\bullet)) \\ F &\longmapsto [F] \end{aligned} \qquad \begin{aligned} q : Z_n(\mathscr{H}om_R(C_\bullet, D_\bullet)) &\longrightarrow \mathrm{H}_n(\mathscr{H}om_R(C_\bullet, D_\bullet)) \\ f &\longmapsto [f] \end{aligned}$$

Then the discussion above and the universal property of group quotient gives us the following commutative diagram

$$\begin{array}{ccc} Z_n(\mathscr{H}om_R(C_\bullet, D_\bullet)) & \xrightarrow{\pi\circ\Phi} & \mathrm{H}_0(\mathscr{H}om_R(C_\bullet[n], D_\bullet)) \\ \downarrow q & \nearrow\swarrow & \uparrow \pi \\ \mathrm{H}_n(\mathscr{H}om_R(C_\bullet, D_\bullet)) & \xleftarrow[q\circ(\Phi)^{-1}]{} & Z_0(\mathscr{H}om_R(C_\bullet[n], D_\bullet)) \end{array}$$

So we have that $\mathrm{H}_n(\mathscr{H}om_R(C_\bullet, D_\bullet)) \cong \mathrm{H}_0(\mathscr{H}om_R(C_\bullet[n], D_\bullet)) \cong [C_\bullet[n], D_\bullet]$.

4. We compute,

$$\begin{aligned} \langle df, c\rangle + (-1)^n \langle f, d^C c\rangle = \langle d^D_{n+q} f - (-1)^n f_{q-1} \circ d^C_q, c\rangle \\ + (-1)^n f_{q-1}(d^C_q(c) = d^D_{n+q} f(c) = d^D \langle f, c\rangle. \end{aligned}$$

So the pairing is a chain map.

5. Let $i + j = n$ and $f \in Hom_R(D, E)$ of degree i, $g \in Hom_R(C, D)$ of degree j. Then, $d(f \otimes g) = (d_i f) \otimes g + (-1)^i f \otimes (d_j g)$, taking composition we have,

$$\begin{aligned} (d_i f)_{l+j} \circ g_l + (-1)^i f_{l+j-1} \circ (d_j g)_l &= d^E_{n+l} f_{l+j} \circ g_l - (-1)^n f_{l+j-1} \circ g_{l-1} \circ d^C_l \\ &\quad + (-1)^i f_{l+j-1} \circ d^D_{j+l} \circ g_l - (-1)^i f_{l+j-1} \circ d^D_{j+l} \circ g_l \\ &= d^E_{n+l} f_{l+j} \circ g_l - (-1)^n f_{l+j-1} \circ g_{l-1} \circ d^C_l \\ &= (d_n(f \circ g))_l \end{aligned}$$

Hence we have the commutativity, where we used $i + j = n$ on the first equation and did the calculation for composition on l^{th} rank.

♢ **Exercice 2. The normalized bar resolution.** The purpose of this exercise is to reduce the bar resolution by killing an "obviously" acyclic subcomplex in the standard bar resolution. Let $F_\bullet$ be the bar resolution, so $F_n = \mathbb{Z}[G^{n+1}]$. Let $D_n \subset F_n$ be generated by the elements $(g_0, \ldots, g_n)$ for which there exists an index i with $g_i = g_{i+1}$.

1. Verify that this defines a subcomplex $D_\bullet$ of $\mathbb{Z}G$-modules. Show that it is generated, as a free $\mathbb{Z}G$-module by the elements $[g_1|\ldots|g_n]$ for which there exists an index i with $g_i = 1$.
2. Show that the contracting homotopy from Sheet 1, Exercise 3, restricts to $D_\bullet$, and therefore induces a contracting homotopy on the normalized chain complex $\overline{F}_\bullet = F_\bullet / D_\bullet$.

3. Identify a basis of $\overline{F}_\bullet$ in degrees $0, 1$, 2, and 3, and the differentials.
4. Let M be a $\mathbb{Z}G$-module. Identify normalized cocycles and coboundaries in $\mathrm{Hom}_{\mathbb{Z}G}(\overline{F}_\bullet; M)$ in degrees 0, 1, and 2.

Solution 2 (by Till and Junda)

1. Let $g = (g_0, \ldots, g_i, g_i, \ldots, g_n) \in D_n$. Then computing $d_n(g)$ yields

$$\sum_{j=0}^{i-1}(-1)^j\partial_j(g) + \underbrace{(-1)^i(g_0, \ldots, g_i, g_{i+2}, \ldots, g_n) + (-1)^{i+1}(g_0, \ldots, g_i, g_{i+2}, \ldots, g_n)}_{=0} + \sum_{j=i+2}^{n}(-1)^j\partial_j(g).$$

The remaining terms all carry the two adjacent and equal coordinates, so that the sum is indeed in D_{n-1}, making $D_\bullet$ a subcomplex. Every $(g_0, \ldots, g_i, g_i, \ldots, g_n) \in D_n$ can be written uniquely as

$$(g_0, \ldots, g_i, g_i, \ldots, g_n) = g_0 \cdot [g_0^{-1}g_1|g_1^{-1}g_2|\ldots|g_{n-1}^{-1}g_n],$$

where the $i+1$-th term will be $g_i^{-1}g_i = 1$. Conversely, by definition all bar elements with a 1 are in D_n and they are all linearly independent (bar elements are linearly independent in general), so they form a basis for D_n and make it a free $\mathbb{Z}G$-module.

2. The homotopy h is defined by adding a 1 in front of all previous coordinates : it preserves the property that some pair of adjacent coordinates are equal. Thus, $h_n(D_n) \subseteq D_{n+1}$ shows that h induces the same homotopy on the normalised chain complexes.

3. According to point 1., one basis for $\overline{F}_n$ is given by

$$\{[g_1|\ldots|g_n] : g_i \neq 1\ \forall i\} \cong (G \setminus \{1\})^n.$$

For $\overline{F}_0$, the generator $[\,]$ is unaffected and forms the usual 1-element basis. For $\overline{F}_1$ we get the basis

$$\{[g] : g \neq 1\} \cong G \setminus \{1\},$$

for $\overline{F}_2$ the basis

$$\{[g|h] : g, h \neq 1\} \cong G \setminus \{1\} \times G \setminus \{1\},$$

and for $\overline{F}_3$ the basis

$$\{[g|h|k] : g, h, k \neq 1\} \cong G \setminus \{1\} \times G \setminus \{1\} \times G \setminus \{1\}.$$

The differentials are as usual, namely

$$d_1([g]) = g[\,]-[\,], \quad d_2([g|h]) = g[h]-[gh]+[g] \ \text{ and } \ d_3([g|h|k]) = g[h|k]-[gh|k]+[g|hk]-[g|h].$$

However, terms might get canceled in the quotient, for instance (there are other combinations)

$$d_2([g|g^{-1}]) = g[g^{-1}] + [g^{-1}] \ \text{ and } \ d_3([g|g^{-1}|g]) = g[g^{-1}|g] - [g|g^{-1}].$$

4. We are now given the cochain complex

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(\overline{F}_0, M) \xrightarrow{d^1} \mathrm{Hom}_{\mathbb{Z}G}(\overline{F}_1, M) \xrightarrow{d^2} \mathrm{Hom}_{\mathbb{Z}G}(\overline{F}_2, M) \xrightarrow{d^3} \mathrm{Hom}_{\mathbb{Z}G}(\overline{F}_3, M) \longrightarrow \cdots$$

Considering the bases found in point 3., we can identify these Hom modules up to isomorphism as below :

$$0 \longrightarrow M \xrightarrow{d^1} \mathrm{Map}_0(G, M) \xrightarrow{d^2} \mathrm{Map}_0(G^2, M) \xrightarrow{d^3} \cdots$$

because of the free $\mathbb{Z}G$-module structure, where we defined

$$\mathrm{Map}_0(G, M) = \{f : G \longrightarrow M : f(1) = 0\} \cong \prod_{g \in G \setminus \{1\}} M[g]$$

and

$$\mathrm{Map}_0(G^2, M) = \{f : G^2 \longrightarrow M : f(g, 1) = 0 = f(1, h)\ \forall g, h \in G\} \cong \prod_{g,h \in G \setminus \{1\}} M[g|h],$$

and so on ; in general we will have

$$\mathrm{Map}_0(G^n, M) = \{f : G^n \longrightarrow M : f(g_1, \dots, g_n) = 0 \text{ if } g_i = 1 \text{ for some } i\} \cong \prod_{g_1,\dots,g_n \in G \setminus \{1\}} M[g_1|\dots|g_n].$$

Further we have $d^i = \mathrm{Hom}(d_i, M)$ given by pre-composition by d_i.
The 0-coboundaries are the image of 0, i.e. are trivial as always. The 0-cocycles are the elements in $\ker d^1$, i.e. elements $m \in M$ such that

$$0 = d^1(m)(g) = ([\,] \mapsto m)(d_1([g])) = ([\,] \mapsto m)(g[\,] - [\,]) = gm - m$$

for all $g \in G$. Thus, this is exactly the invariants M^G and because the 0-coboundaries are trivial, $H^0(G, M) \cong M^G$.
The 1-coboundaries are elements in the image of d^1, i.e. maps of the form

$$f(g) = d^1(m)(g[\,]) = gm - m$$

for some $m \in M$ (notice that this forces $f(1) = 0$ as needed). The 1-cocyles are the $\mathbb{Z}G$-homomorphisms $F \in \mathrm{Hom}(\overline{F}_1, M)$, extended linearly from the data of some $f \in \mathrm{Map}_0(G, M)$, verifying

$$0 = (d^2(F))(g, h) = F(d_2([g|h])) = g \cdot f(h) - f(gh) + f(g),$$

i.e. the maps verifying $f(gh) = g \cdot f(h) + f(g)$, for all $g, h \in G$ (notice that setting g or h to 1 still works because we forced $f(1) = 0$). Maps of this form are called *derivations* and we write their group $\mathrm{Der}(G, M) = \ker d^2$. The 1-coboundaries are called *principal derivations*, and form a subgroup $\mathrm{Prin}(G, M) \leq \mathrm{Der}(G, M)$, leading to the result

$$H^1(G, M) = \mathrm{Der}(G, M)/\,\mathrm{Prin}(G, M).$$

The 2-coboundaries are the maps in the image of d^2, i.e. maps verifying

$$F(g, h) = g \cdot f(h) - f(gh) + f(g) \quad \text{for all} \quad g, h \in G,$$

for some $f \in \mathrm{Map}_0(G, M)$. Finally, the 2-cocyles are the $\mathbb{Z}G$-homomorphisms $F \in \mathrm{Hom}(\overline{F}_2, M)$, extended linearly from the data of some $f \in \mathrm{Map}_0(G^2, M)$, verifying

$$0 = d^3(F)(g, h, k) = \cdots = g \cdot f(h, k) - f(gh, k) + f(g, hk) - f(g, h).$$

♢ **Exercice 3. Extension of scalars.** Let $\alpha\colon R \to S$ be a ring homomorphism. We will write M for an R-module and N for an S-module.

1. **Extension of scalars.** Show that $S \otimes_R M$ is an S-module and the map $M \to S \otimes_R M$ defined by $m \mapsto 1 \otimes m$ is an R-module map.
2. **Universal property.** Any R-module map $f\colon M \to N$ factors through the extension of scalars map from point 1.
3. Apply Point 2 to the identity to obtain a *canonical* S-module map $S \otimes_R N \to N$ and show its surjectivity.
4. **Co-extension of scalars.** Dualize briefly to obtain a canonical injective S-module map $N \to \mathrm{Hom}_R(S, N)$.
5. Identify the extension and co-extension of scalars associated to the inclusion $\alpha\colon \mathbb{Z}H \to \mathbb{Z}G$.

Solution 3. (by Eliot and Damien)

1. We define the S-module product in the following way : for $t \in S, s \otimes_R m \in S \otimes_R M$, $t \cdot (s \otimes_R m) := ts \otimes_R m$ and extend distributively on general element of $S \otimes_R M$.

   We still have to check that the action of S is associative with the product of the ring S. It arises naturally from the associativity of S : for $t_1, t_2 \in S$, $(t_1 t_2) \cdot (s \otimes_R m) = ((t_1 t_2)s) \otimes_R m = (t_1(t_2 s)) \otimes_R m = t_1 \cdot (t_2 s \otimes_R m)$. Also we have that $1 \cdot (s \otimes_R m) = (1 \cdot s) \otimes_R m = s \otimes_R m$.

   We want to show that the map $\varphi : M \to S \otimes_R M$ sending m to $1 \otimes_R m$ is a R-module map. For $r \in R, m \in M : \varphi(rm) = 1 \otimes_R rm = \alpha(r) \otimes_R m = r \cdot_R (1 \otimes_R m) = r \cdot_R \varphi(m)$ where $\cdot_R$ is the scalar multiplication on the $R-$module $S \otimes_R M$.
2. Define a map $\psi_f : S \otimes_R M \to N$ by $\psi_f(s \otimes_R m) := s \cdot_N f(m)$ where $\cdot_N$ is the scalar multiplication in the S- module N. It is well defined since f is a R-module map. Then by definition, $f(m) = \psi_f \circ \varphi(m)$.

   We want to check that it is a morphism of R-module. Let $r \in R, s \otimes_R m \in S \otimes_R M$: $\psi_f(r \cdot_R (s \otimes_R m)) = \psi_f((\alpha(r) \cdot s) \otimes_R m) = (\alpha(r) \cdot s) \cdot_N f(m) = r \cdot_N (s \cdot_N f(m)) = r \cdot_N \psi_f(s \otimes_R m)$

   Here we abuse a bit of the notation since N is an S-module but also a R-module with the multiplication $r \cdot n := \alpha(r) \cdot n$.
3. We get a map $\psi_{id} : S \otimes_R N \to N : s \otimes_R n \to s \cdot n$, it is clear that it is a S- module map. The map is surjective since for any $n \in N, \psi_{id}(1 \otimes_R n) = n$.
4. Let's dualize the previous constructions. First of all the dual object of $S \otimes_R M$ is $Hom_R(S, M)$, it is also an S-module, and we have a R-module map $\phi : Hom_R(S, M) \to M$ given by $g \mapsto g(1)$. Now observe that for any R-module map $f : N \to M$, it factors through the co-extension of scalars :

   $$N \xrightarrow{\psi} Hom_R(S, M) \xrightarrow{\phi} M$$

   where $\psi(n)(s) = f(s \cdot n)$, so that $f = \phi \circ \psi$. We can apply this construction to $id : N \to N$, and we get a an S-module map $N \to Hom_R(S, N)$ given by $n \mapsto g_n$, where $g_n(s) = s \cdot n$. And this map is injective because if $g_n = g_{n'}$, we have in particular $n = g_n(1) = g_{n'}(1) = n'$.

5. Let H be a subgroup of G, and $\alpha : \mathbb{Z}H \to \mathbb{Z}G$ be the inclusion. Then the extension of scalars sends the H-module M to the G-module $\mathbb{Z}G \otimes_{\mathbb{Z}H} M$. So the extension of scalars is the induction from H to G. And the co-extension of scalars sends the H-module M to the G-module $Hom_{\mathbb{Z}H}(\mathbb{Z}G, M)$. So co-extension of scalars is the coinduction from H to G.